Aronszajn's Criterion for Euclidean Space

R.D. Arthan*

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Abstract

We give a simple proof of a characterization of euclidean space due to Aronszajn and derive a well-known characterization due to Jordan & von Neumann as a corollary.

A norm $||\_||$ on a vector space V is *euclidean* if there is an inner product $\langle \_, \_\rangle$ on V such that $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$. Characterizations of euclidean normed spaces abound. Amir [1] surveys some 350 characterizations, starting with a well-known classic of Jordan & von Neumann [3]: a norm is euclidean iff it satisfies the *parallelogram identity*:

$$||\mathbf{v}+\mathbf{w}|| \;\; = \;\; \sqrt{2||\mathbf{v}||^2 + 2||\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2}$$

Aronszajn proved that the algebraic details of this identity are mostly irrelevant: if the norms of two sides and one diagonal of a parallelogram determine the norm of the other diagonal then the norm is euclidean. Formally, Aronszajn's criterion is the following property, as illustrated in figure 1(a).

$$\begin{array}{c}\forall \mathbf{v}_1\ \mathbf{w}_1\ \mathbf{v}_2\ \mathbf{w}_2\cdot\ ||\mathbf{v}_1|| = ||\mathbf{v}_2|| \wedge ||\mathbf{w}_1|| = ||\mathbf{w}_2|| \wedge ||\mathbf{v}_1 - \mathbf{w}_1|| = ||\mathbf{v}_2 - \mathbf{w}_2|| \\ \Rightarrow ||\mathbf{v}_1 + \mathbf{w}_1|| = ||\mathbf{v}_2 + \mathbf{w}_2||.\end{array}$$

Aronszajn's announcement of this characterization [2] does not give a proof. Amir's proof forms part of a long chain of interrelated results. In this note we give a short, self-contained proof of the theorem and derive the Jordan-von Neumann theorem as a corollary. We begin with a lemma showing that the Aronszajn criterion ensures a useful supply of isometries. Figure 1(b) illustrates the parallelograms that feature in the proof.

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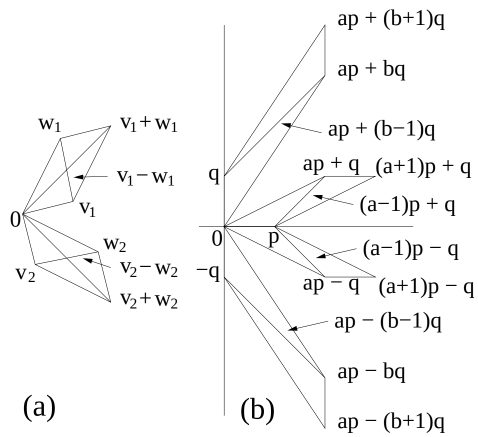

Figure 1: Some parallelograms (a bird and an imp).

Lemma 1 *Let V be a 2-dimensional normed space satisfying the Aronszajn criterion. If $\mathbf{0} \neq \mathbf{p}, \mathbf{q} \in V$ and $||\mathbf{p}+\mathbf{q}|| = ||\mathbf{p}-\mathbf{q}||$, then there is a linear isometry $\mu : V \to V$ such that $\mu(\mathbf{p}) = \mathbf{p}$ and $\mu(\mathbf{q}) = -\mathbf{q}$.*

Proof: The equation $||\mathbf{p}-\mathbf{q}|| = ||\mathbf{p}+\mathbf{q}||$ cannot hold if the non-zero vectors $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent, so $\mathbf{p}$ and $\mathbf{q}$ form a basis for V and $\mu(a\mathbf{p}+b\mathbf{q}) = a\mathbf{p}-b\mathbf{q}$ will define the required isometry provided the following holds for all $a, b \in \mathbb{R}$.

$$||a\mathbf{p}+b\mathbf{q}|| \quad = \quad ||a\mathbf{p}-b\mathbf{q}|| \qquad (*)$$

(*) is trivial when $a = 0$ or $b = 0$ and is true by assumption when $a = b = 1$. The instances of the Aronszajn criterion displayed below hold in V by assumption. As the antecedent of the implication may be assumed by induction for integer $a > 1$, we conclude that (*) holds for $a \in \mathbb{N}$ and $b = 1$.

$$\begin{array}{rrcl} & ||a\mathbf{p}+\mathbf{q}|| & = & ||a\mathbf{p}-\mathbf{q}|| \\ \wedge & ||\mathbf{p}|| & = & ||\mathbf{p}|| \\ \wedge & ||(a-1)\mathbf{p}+\mathbf{q}|| & = & ||(a-1)\mathbf{p}-\mathbf{q}|| \\ \Rightarrow & ||(a+1)\mathbf{p}+\mathbf{q}|| & = & ||(a+1)\mathbf{p}-\mathbf{q}|| \end{array}$$

This gives the base case for an induction on b showing that (*) holds for $a, b \in \mathbb{N}$ using the following instances of the Aronszajn criterion.

$$\begin{array}{rrcl} & ||a\mathbf{p}+b\mathbf{q}|| & = & ||a\mathbf{p}-b\mathbf{q}|| \\ \wedge & ||\mathbf{q}|| & = & ||-\mathbf{q}|| \\ \wedge & ||a\mathbf{p}+(b-1)\mathbf{q}|| & = & ||a\mathbf{p}-(b-1)\mathbf{q}|| \\ \Rightarrow & ||a\mathbf{p}+(b+1)\mathbf{q}|| & = & ||a\mathbf{p}-(b+1)\mathbf{q}|| \end{array}$$

By symmetry, (*) holds for all $a, b \in \mathbb{Z}$; using $||(j/k)\mathbf{p} + (m/n)\mathbf{q}|| = |1/(kn)| \cdot ||jn\mathbf{p} + km\mathbf{q}||$, we find that (*) holds for all $a, b \in \mathbb{Q}$; finally, by continuity, (*) holds for all $a, b \in \mathbb{R}$. ■

The next lemma shows that the conclusion of lemma 1 characterizes euclidean space amongst 2-dimensional normed spaces.

Lemma 2 *Let V be a 2-dimensional normed space such that if $\mathbf{0} \neq \mathbf{p}, \mathbf{q} \in V$ and $||\mathbf{p} + \mathbf{q}|| = ||\mathbf{p} - \mathbf{q}||$, then there is a linear isometry $\mu : V \to V$ with $\mu(\mathbf{p}) = \mathbf{p}$ and $\mu(\mathbf{q}) = -\mathbf{q}$. Then V is euclidean.*

Proof: Let $\mathbf{e}_1 \in V$ be a unit vector. As $\mathbf{x}$ traverses an arc of the unit circle from $\mathbf{e}_1$ to $-\mathbf{e}_1$, $||\mathbf{e}_1 + \mathbf{x}|| - ||\mathbf{e}_1 - \mathbf{x}||$ varies continuously from 2 to -2 and hence is 0 for some $\mathbf{x}$. Let $\mathbf{e}_2$ be such an $\mathbf{x}$. Take euclidean coordinates with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$ and so fix a euclidean norm $||\_||_e$ on V together with the associated notions of angle, rotation and reflection in a line. The condition $||\mathbf{p} + \mathbf{q}|| = ||\mathbf{p} - \mathbf{q}||$ is satisfied both for $\mathbf{p} = \mathbf{e}_1$, $\mathbf{q} = \mathbf{e}_2$ and for $\mathbf{p} = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{q} = \mathbf{e}_1 - \mathbf{e}_2$, and so, by assumption, the reflections in the x-axis and in the line $x = y$ are both V-isometries, and hence so is their composite, a rotation ρ through a right angle. Let $\mathbf{u}$ be any unit vector in V, let $\mathbf{p}$ bisect the angle between $\mathbf{u}$ and $\mathbf{e}_1$ and let $\mathbf{q} = \rho(\mathbf{p})$. We then have:

$$||\mathbf{p} + \mathbf{q}|| = ||\rho(\mathbf{p} + \mathbf{q})|| = ||\rho(\mathbf{p}) + \rho(\mathbf{q})|| = ||\mathbf{q} - \mathbf{p}|| = ||\mathbf{p} - \mathbf{q}||$$

and so our assumptions imply that the reflection μ in the bisector of the angle between $\mathbf{u}$ and $\mathbf{e}_1$ is a V-isometry. Thus $\mathbf{u}$ and $\mu(\mathbf{e}_1)$ have the same direction and the same V-norm, and so $\mathbf{u} = \mu(\mathbf{e}_1)$. Hence $||\mathbf{u}||_e = ||\mu(\mathbf{e}_1)||_e = ||\mathbf{e}_1||_e = 1$, for every unit vector $\mathbf{u}$ of V, and V is therefore euclidean. ■

To lift results from 2 dimensions to arbitrary dimensions we use the following lemma, which Amir proves using the Jordan-von Neumann theorem, as did Jordan & von Neumann. We give a geometric proof.

Lemma 3 *If every 2-dimensional subspace of a normed space V is euclidean, then V is euclidean.*

Proof: Let d be the (possibly infinite) dimension of V. If $d = 0, 1$ or 2, the theorem is trivially true, so we may assume $d \geq 3$. If V is euclidean, then easy algebra shows that the inner product $\langle \_, \_ \rangle$ must be given by:

$$\langle \mathbf{v}, \mathbf{w} \rangle = (||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v}||^2 - ||\mathbf{w}||^2)/2.$$

So V is euclidean iff the function $\langle \mathbf{v}, \mathbf{w} \rangle$ defined by the above equation satisfies the axioms for an inner product. These axioms can be expressed using just 3 vector variables, and so they hold in V iff they hold in every subspace of V of dimension at most 3. Thus we may assume $d = 3$.

Let U be a 2-dimensional and so euclidean subspace of V; let $\mathbf{e}_1$ and $\mathbf{e}_2$ be orthogonal unit vectors in U; and let $\mathbf{e}_3$ lie in the intersection of the unit sphere S_V and a supporting plane of S_V, say $P = \mathbf{e}_3 + U$, parallel to U. For $\theta \in [0, \pi)$, let $\mathbf{p}_\theta$ be the unit vector $\cos\theta \cdot \mathbf{e}_1 + \sin\theta \cdot \mathbf{e}_2$ in U and let W_θ be the subspace of V spanned by $\mathbf{p}_\theta$ and $\mathbf{e}_3$. W_θ is 2-dimensional and so euclidean. The line $P \cap W_\theta$ is parallel to $\mathbf{p}_\theta$ and meets S_{W_θ} at $\mathbf{e}_3$. Hence as P supports S_V, $P \cap W_\theta$ must support the unit circle S_{W_θ} of W_θ. It follows that $\mathbf{e}_3$ is a unit vector orthogonal to $\mathbf{p}_\theta$ in W_θ and that $S_{W_\theta} = \{\sin\phi \cdot \mathbf{p}_\theta + \cos\phi \cdot \mathbf{e}_3 \mid \phi \in [0, 2\pi)\}$. But S_V is the union of the S_{W_θ}, so we have:

$$S_V = \{\sin\phi \cdot \cos\theta \cdot \mathbf{e}_1 + \sin\phi \cdot \sin\theta \cdot \mathbf{e}_2 + \cos\phi \cdot \mathbf{e}_3 \mid \theta \in [0, \pi), \phi \in [0, 2\pi)\}$$

I.e., the unit sphere of V is a euclidean sphere, and V must be euclidean. ■

Theorem 4 (Aronszajn; Jordan & von Neumann) *If V is a normed space, then the following are equivalent:*

(i) the Aronszajn criterion holds in V;
(ii) V is euclidean;
(iii) the parallelogram identity holds in V.

Proof: [*(i)* $\Rightarrow$ *(ii)*]: this is immediate from lemmas 1, 2 and 3;
[*(ii)* $\Rightarrow$ *(iii)*]: easy algebra shows that with $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ the parallelogram identity holds in an inner product space;
[*(iii)* $\Rightarrow$ *(i)*]: the parallelogram identity clearly implies the Aronszajn criterion. ■

References

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[3] P. Jordan and J. von Neumann. On inner products in linear, metric spaces. *Annals of Mathematics (2nd series)*, 36(3):719–723, 1935.